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Enumerating 4×5 and 5×6 double Youden rectangles

C. Christofi

Institute of Mathematics and Statistics, Cornwallis Building, The University, Canterbury, Kent CT2 7NF, UK

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Abstract

Double Youden rectangles of sizes 4×5 and 5×6 are classified into species, each comprising 1, 2 or 4 transformation sets. The species and transformation sets are fully enumerated. For size 4×5 there are 2 species, each comprising a single transformation set; for size 5×6 there are 14 species and 38 transformation sets. The full automorphism groups are found for the members of each species.

1. Introduction

A double Youden rectangle (DYR) of size $k \times v$ was defined by Bailey [1, p. 40] to be an arrangement of kv ordered pairs x, y in k rows and v columns ($k<v$) such that

(i) each value x is drawn from a set S of v elements;
(ii) each value y is drawn from a set T of k elements;
(iii) each element from S occurs exactly once in each row and no more than once per column;
(iv) each element from T occurs exactly once in each column and either n or $n+1$ times in each row, where n is the integral part of v/k;
(v) each element from S is paired exactly once with each element from T;
(vi) each pair of elements from S occurs together in exactly λ columns, where $\lambda = k(k-1)/(v-1)$, i.e. the sets of elements of S in the columns are the blocks of a symmetric balanced incomplete block design (SBIBD or a symmetric 2-design) with parameters $\{v, k, \lambda\}$;
(vii) if n occurrences of each element from T are removed from each row, leaving $m = v - nk$ elements from T in each row, then (a) the remaining sets of elements

Correspondence to: C. Christofi, Institute of Mathematics and Statistics, University of Kent, Cornwallis Building, Canterbury, Kent CT2 7NF, UK.

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of T in the rows are the blocks of a SBIBD with parameters $\{k, m, \mu\}$, where $\mu = m(m-1)/(k-1)$, or else (b) $m=1$.

DYRs discussed in this paper are of size $(v-1)\times v$. Thus, $n=1$ in (iv), the SBIBD in (vi) is trivial, with each block having every treatment except one, and in (vii) $m=1$. Removing the elements of T from a DYR leaves a Youden square.

Since the appearance of the first DYR [4], work on construction of DYRs has exceeded work on their enumeration. DYRs of size $(v-1)\times v$ appeared in Preece [11, 12]. Preece [11] showed how to obtain what he called a 'structurally different' 4×5 DYR from the one obtained by Clarke [4] and also showed that there are at least 10 'structurally different' 5×6 DYRs. Clarke [5] gave DYRs of size 4×7, while Preece [12] gave examples of DYRs of size 6×7, 7×15, 11×23 and 19×39. Preece [13] has reviewed the literature on DYRs and given examples of DYRs of size 5×11. Hedeyat et al. [8] gave a construction for $(v-1)\times v$ DYRs, based on pairs of orthogonal latin squares with a common transversal.

In the first enumerative study of DYRs, Clarke [5] enumerated DYRs where the elements of the set S are arranged in a $(v-1)\times v$ Youden square. To obtain $(v-1)\times v$ Youden squares he successively omitted each row from each of the standard $v\times v$ latin squares given by Fisher and Yates [6], $v=5,6$. For each of the resulting Youden squares he then searched exhaustively for all allocations of the set T that, superimposed on the given Youden square, produced an arrangement satisfying conditions (i)–(vii) above. Clarke termed each such arrangement a 'solution', and classified these 'solutions' according to the location of the duplicated element belonging to set T. Clarke obtained 10 such 'solutions' of size 4×5 and 114 'solutions' for the 5×6 DYRs.

This paper reports a full enumeration and classification of nonisomorphic $(v-1)\times v$ DYRs with $v=5$ and $v=6$. By scrutiny of representative members of classes of 6×6 latin squares [7], an additional 60 'solutions' for $v=6$ have been obtained. However, before we can describe our approach in detail, the notions of isomorphism and automorphism in relation to DYRs must be introduced.

2. Transformations and invariants of DYRs

Let the sets of rows and columns of a DYR be denoted by P and Q respectively. Then P, Q, S and T can be considered as the four 'constraints' of a DYR ω. Let Ω be a set of DYRs

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n\}$$

and let $\alpha = (p, q, s, t)$ be a quadruple where p is a permutation of rows, P, q is a permutation of columns Q, s is a permutation of the elements of S and t is a permutation of the elements of T. If there exists an α such that $\alpha : \omega_i \to \omega_j$, then ω_i and ω_j are said to be *isomorphic* (or in the terminology of Preece [13] 'equivalent') and α is said to be an *isomorphism* from ω_i to ω_j.

The set Ω is said to be a *transformation set* if and only if

$$\forall \omega_i, \omega_j \in \Omega \ \exists \alpha : \omega_i \to \omega_j.$$

Preece [13] showed that, given one DYR, we have in fact four 'adjugate' DYRs to consider. Let v represent the operation of interchanging the roles of the constraints S and Q, ϕ the operation of interchanging the roles of constraints P and T, and $(v\phi)$ $(=(\phi v))$ the operation of v followed by that of ϕ. Then the adjugates ω^*, $^*\omega$ and $^*\omega^*$ of ω can be obtained by operating on ω, where $v(\omega)=\omega^*$, $\phi(\omega)={}^*\omega$ and $v\phi(\omega)={}^*\omega^*$. Indeed, $\omega, \omega^*, {}^*\omega$ and $^*\omega^*$ may or may not all be isomorphic to each other, the possibilities being as follows: (i) none of the four adjugates is isomorphic to any of the others; (ii) ω is isomorphic to one of its three adjugates but not to either of the other two, which are then isomorphic to one another; (iii) all four adjugates are isomorphic to one another.

We say that a set of Ω of DYRs forms a *species* if and only if

$$\forall \omega_i, \omega_j \in \Omega \ \exists \alpha : \omega_i \to \omega_j \quad \text{and} \quad \forall \omega_i \in \Omega,\ \omega_i^*, {}^*\omega_i, {}^*\omega_i^* \text{ are contained in } \Omega,$$

i.e.

$$\Omega = \{\omega_1, \omega_1^*, {}^*\omega_1, {}^*\omega_1^*, \omega_2, \omega_2^*, {}^*\omega_2, {}^*\omega_2^*, \ldots, \omega_n, \omega_n^*, {}^*\omega_n, {}^*\omega_n^*\}.$$

Hence, a species can contain 1, 2 or 4 transformation sets, the number depending on isomorphisms between ω_i and its adjugates. An *automorphism* is an isomorphism of a DYR onto itself, i.e. there exists an α such that $\alpha : \omega_i \to \omega_i$. Then α is said to be an automorphism of ω_i. The collection of all automorphisms of ω forms a group under composition called the *full automorphism group* and is denoted by $\operatorname{Aut}\omega$.

3. Enumeration and classification of $(v-1) \times v$ DYRs

Our method begins by examining the approach used by Clarke [5]. In agreement with Clarke, we found 10 'solutions' (DYRs) of size 4×5. However, by considering certain 6×6 standard latin squares given in [6, 7], we obtained an additional 60 'solutions' (DYRs) of size 5×6.

The enumeration of DYRs was aided by work linking design isomorphism and graph isomorphism, in particular the work of McKay [9]. Using the usual definition of design, McKay and Stanton [10] and Bohm and Santolini [2] showed that two designs are isomorphic if their respective graphs are isomorphic. An isomorphism testing program of McKay [9] was used to sift for isomorphic 'solutions' among the 10 'solutions' of size 4×5 and 174 'solutions' of size 5×6. We classified DYRs into transformation sets and species, and found the full automorphism group of each DYR. All the full automorphism groups are isomorphic to known groups.

Table 1
Isomorphisms amoung Clarke's 10 'solutions' of size 4×5

Solution	Isomorphic mate	Isomorphism S	T	P	Q
1	6	(DE)	(cd)	(34)	(45)
2	3	(BC)(CD)	(cd)	(34)	(23)(45)
2	4	(BDC)	—	—	(243)
2	5	(BEC)	(cd)	(34)	(253)
2	7	(DE)	(cd)	(34)	(45)
2	8	(BC)	—	—	(23)
2	9	(BDEC)	(cd)	(34)	(2453)
2	10	(BEDC)	—	—	(2543)

Table 2
Representatives of 4×5 DYRs and generator(s) of their automorphism

Solution	Generator(s) of Aut ω S	T	P	Q
1				
Ba Aa Eb Cc Dd				
Cd Dc Ad Ea Bb	(CDE)	(bcd)	(234)	(345)
Db Ed Bc Ab Ca	(BC)(DE)	(ad)(bc)	(12)(34)	(23)(45)
Ec Cb Da Bd Ac				
2				
Aa Ba Cb Dc Ed				
Cc Db Ad Ea Bb	(CDE)	(bcd)	(234)	(345)
Dd Ec Bc Ab Ca				
Eb Cd Da Bd Ac				
6				
Ba Aa Db Ec Cd				
Cc Ed Ac Bb Da	(CDE)	(bcd)	(234)	(435)
Dd Cb Ea Ad Bc	(BC)(DE)	(ac)(bd)	(12)(34)	(23)(45)
Eb Dc Bd Ca Ab				

4. Transformation sets, species and automorphism groups of DYRs

Table 1 shows which of the 10 'solutions' of 4×5 DYRs are isomorphic to one another. Solution 1 is isomorphic to solution 6 and solution 2 is isomorphic to all the remaining 'solutions'. We therefore have two nonisomorphic 4×5 DYRs and thus 2 transformation sets. Since solution 1 is not an adjugate of solution 2, the 2 transformation sets each belong to a different species. Table 2 lists representatives (solutions 1 and 2) for the two transformation sets and generators for their automorhism groups. Solution 6 is also given and can be used to show how isomorphisms in Table 1 work. Solutions 1 and 6 have full automorphism groups of order 12, while the remaining 'solutions' have full automorphism groups of order 3. Indeed, 'solutions' (2, 3, 4, 5, 7, 8, 9, 10) have automorphism groups isomorphic to the cyclic group of order 3 (C_3). More interesting, however, are the full automorphism groups of 'solutions' 1 and 6, both of which have full automorphism groups isomorphic to the alternating group of degree 4 (A_4).

Table 3
Representatives of the 14 species of 5 × 6 DYRs

1

Aa	Bb	Cc	Dd	Ee	Fa
Bc	Ad	Eb	Ca	Fc	De
Db	Ea	Fd	Be	Cd	Ac
Ed	Dc	Ae	Fb	Ba	Cb
Fe	Ce	Da	Ec	Ab	Bd

2

Aa	Bb	Cc	Dd	Ee	Fc
Bd	Ac	Ea	Cb	Fb	De
Dc	Ed	Fd	Be	Ca	Ab
Eb	Da	Ae	Fa	Bc	Cd
Fe	Ce	Db	Ec	Ad	Ba

3

Aa	Bb	Cc	Dd	Ea	Fe
Bc	Ae	Ed	Ca	Fb	Dc
Db	Ec	Fa	Be	Ce	Ad
Ee	Da	Ab	Fc	Bd	Cb
Fd	Cd	De	Eb	Ac	Ba

4

Aa	Bb	Cc	Dd	Ec	Fe
Be	Ac	Ed	Cb	Fa	Db
Dc	Ea	Fb	Ba	Ce	Ad
Eb	De	Ae	Fc	Bd	Ca
Fd	Cd	Da	Ee	Ab	Bc

5

Ba	Aa	Eb	Fc	Cd	De
Cc	Fb	Ae	Be	Da	Ed
Db	Ec	Bd	Ad	Fe	Ca
Ee	Dd	Fa	Cb	Bc	Ac
Fd	Ce	Dc	Ea	Ab	Bb

6

Ba	Aa	Eb	Fc	Cd	De
Cb	Fe	Ac	Bd	Da	Ec
Dc	Ed	Be	Ab	Fb	Ca
Ee	Db	Fa	Ce	Bc	Ad
Fd	Cc	Dd	Ea	Ae	Bb

7

Ba	Aa	Eb	Fc	Cd	De
Cc	Fd	Ae	Bb	Da	Ec
Db	Ee	Bc	Ad	Fb	Ca
Ed	Dc	Fa	Ce	Be	Ab
Fe	Cb	Dd	Ea	Ac	Bd

8

Ba	Aa	Eb	Fc	Cd	De
Cc	Fb	Be	Ae	Da	Ed
Db	Ec	Ad	Bd	Fe	Ca
Ee	Dd	Fa	Cb	Bc	Ac
Fd	Ce	Dc	Ea	Ab	Bb

9

Ba	Aa	Eb	Cc	Fd	De
Ce	Fc	Be	Ab	Da	Ed
Dd	Ee	Fa	Bd	Cb	Ac
Ec	Db	Ad	Fe	Bc	Ca
Fb	Cd	Dc	Ea	Ae	Bb

10

Ba	Aa	Eb	Fc	Cd	De
Ce	Fd	Bd	Ab	Dc	Ea
Dd	Ee	Ac	Be	Fa	Cb
Ec	Db	Fe	Ca	Bb	Ad
Fb	Cc	Da	Ed	Ae	Bc

11

Aa	Ba	Cb	Dc	Ed	Fe
Ce	Fc	Bc	Ad	Db	Ea
Dd	Eb	Ae	Bb	Fa	Cc
Ec	De	Fd	Ca	Be	Ab
Fb	Cd	Da	Ee	Ac	Bd

12

Ba	Aa	Eb	Cc	Fd	De
Cd	Fb	Bd	Ae	Dc	Ea
Db	Ec	Fe	Bb	Ca	Ad
Ee	Dd	Ac	Fa	Be	Cb
Fc	Ce	Da	Ed	Ab	Bc

13

Ba	Aa	Fb	Ec	Dd	Ce
Cc	Db	Ae	Bb	Fa	Ed
De	Fc	Ea	Ad	Cb	Bc
Eb	Cd	Bd	Fe	Ac	Da
Fd	Ee	Dc	Ca	Be	Ab

14

Ba	Aa	Eb	Cc	Fd	De
Ce	Ed	Ae	Fa	Db	Bc
Dd	Cb	Fc	Ad	Be	Ea
Ec	Fe	Da	Bb	Ac	Cd
Fb	Dc	Bd	Ee	Ca	Ab

Table 4
Generator(s) and group order of 5 × 6 DYRs

Representative ω	Group order	Generator(s) of Aut ω P	Q	S	T	No. of transformation sets within ω
1	2	(12)(34)	(34)(56)	(AB)(DE)	(ac)(bd)	4
2	2	(12)(34)	(34)(56)	(AB)(DE)	(ad)(bc)	4
3	2	(12)(34)	(34)(56)	(AB)(DE)	(ac)(be)	$2(v\phi)$
4	2	(12)(34)	(34)(56)	(AB)(DE)	(ae)(bc)	$2(v\phi)$
5	2	(23)(45)	(34)(56)	(CD)(EF)	(bc)(de)	$2(\phi)$
6	2	(23)(45)	(34)(56)	(CD)(EF)	(bc)(de)	$2(v\phi)$
7	2	(23)(45)	(34)(56)	(CD)(EF)	(bc)(de)	$2(v\phi)$
8	4	(23)(45)	(34)(56)	(CD)(EF)	(bc)(de)	4
		(24)(35)	(35)(46)	(CE)(DF)	(bd)(ce)	
9	4	(2354)	(3465)	(CDFE)	(bcde)	4
10	4	(23)(45)	(34)(56)	(CD)(EF)	(bc)(de)	$2(v\phi)$
		(24)(35)	(35)(46)	(CE)(DF)	(bd)(ce)	
11	4	(23)(45)	(34)(56)	(CD)(EF)	(bc)(de)	$2(v\phi)$
		(24)(35)	(35)(46)	(CE)(DF)	(bd)(ce)	
12	4	(2354)	(3465)	(CDFE)	(bcde)	$2(v\phi)$
13	20	(2354)	(3465)	(CDFE)	(bced)	4
		(12)(45)	(14)(35)	(AD)(CE)	(ab)(de)	
14	20	(2354)	(3465)	(CDFE)	(bced)	$2(v\phi)$
		(15432)	(26543)	(BFEDC)	(adbec)	

5. 5 × 6 double Youden rectangles

The 174 'solutions' (DYRs) of size 5×6 reduce to 38 nonisomorphic DYRs and consequently 38 transformation sets. The 38 transformation sets fall into 14 species. We find that 5×6 DYRs can have full automorphism groups of order $(v-4)$, $(v-2)$ and $(v-2)(v-1)$. Since there is only one isomorphism class of groups of order 2 ($C_2 \equiv D_1 \equiv S_1$), DYRs with full automorphism groups of order 2 have full automorphism groups isomorphic to C_2. DYRs with full automorphism groups of order 4 have full automorphism groups isomorphic to each of the possible groups of order 4, namely D_2 ($\equiv C_2 \times C_2$) or C_4. The full automorphism groups of order 20 are isomorphic to one of the five *metacyclic* groups of order 20. For details of metacyclic groups, see Carmichael [3]. Table 3 gives the representatives of all 14 species. For each representative, Table 4 gives the order of automorphism group and the number of transformation sets it contains. If a species contains 2 transformation sets, the operation stabilising the transformations set is given.

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